Day 1: General

Topic: The Tower of Hanoi

The Tower of Hanoi is a classic mathematical puzzle that shows the power of **recursion**. The setup consists of three rods and a stack of disks of distinct sizes. Initially, the disks are stacked on the first rod in increasing order of size, with the smallest at the top and the largest at the bottom.

The Rules:

1. You may only move one disk at a time.
2. You must take the top disk from one stack and place it onto another stack.
3. You cannot place a larger disk onto a smaller disk.

The Recursive Strategy: To solve this, imagine you need to move the bottom-most disk (the largest one) to the target rod. You cannot move it until all the other smaller disks sitting on top of it are moved out of the way. Therefore, the strategy for moving a stack of n disks is:

- **Step 1:** Move the top $n-1$ disks from the Source Rod to the Spare Rod.
- **Step 2:** Move the largest disk (the n-th disk) from the Source Rod to the Target Rod.
- **Step 3:** Move the $n-1$ disks from the Spare Rod to the Target Rod (on top of the large disk).

This breaks a large problem down into smaller, identical versions of itself.

Questions

1. **Short Answer:** What is the minimum number of moves required to solve the puzzle with 3 disks?
2. **Short Answer:** What is the minimum number of moves required to solve the puzzle with 5 disks, given the constraint that you can only move a disk to an adjacent rod?
3. **Proof:** Let T_n be the minimum number of moves required for n disks. Find a closed formula to find T_n in terms of n and prove.

Day 2: Algebra

Topic: The AM-GM Inequality

In algebra, we often want to find the minimum or maximum value of an equation. Usually, Calculus (derivatives) is used to do this. However, the **Arithmetic Mean-Geometric Mean (AM-GM)** inequality allows you to optimize functions using only algebra.

The inequality states that for any set of non-negative real numbers, the arithmetic mean (the average) is always greater than or equal to the geometric mean.

For two numbers a and b:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Why is this useful? If the product ab is a constant, the sum $a+b$ has a specific minimum value. This is extremely helpful for equations involving reciprocals, such as $x + \frac{1}{x}$, because the x terms cancel out when multiplied, leaving a constant.

Note that the "equals" sign in $\geq$ only holds true when the variables are equal ($a = b$). This fact helps us pinpoint exactly what value of x gives the minimum.

Questions

1. **Short Answer:** Find the minimum value of the expression $x + \frac{9}{x}$ for $x > 0$.

2. **Short Answer:** A rectangular box with a lid has volume 144 cubic inches. The length, width, and height are positive real numbers. The base and lid cost 2 cents per square inch, and the four sides cost 3 cents per square inch. What is the minimum possible cost, in cents?

3. **Proof:** Let a and b be positive real numbers. Prove the AM-GM inequality, and determine all cases in which equality holds.

Day 3: Geometry

Topic: Pythagorean Triples

A Pythagorean triple is a set of three integers a, b, c that satisfy the right-triangle theorem $a^2+b^2 = c^2$. While many students memorize $(3, 4, 5)$ or $(5, 12, 13)$, there is a simple algorithm to generate a triple for **any** odd number greater than 1.

Questions

1. **Short Answer:** How many right triangles of integral side lengths can be formed with a side length of 2, 24, or 2026?

2. **Short Answer:** Let S be a semi-circle with diameter $AB = 25$. Points C and D are placed on the arc of S such that the chord lengths are $AC = 15$ and $BD = 7$. The chords AD and BC intersect at a point P. Let M be the foot of the perpendicular from P to the diameter AB. Find the length of the segment PM.

3. **Proof:** Let x be an odd integer ($x > 1$). Prove that there will always be a Pythagorean triple containing x as the shortest side length, and provide the Pythagorean triple in terms of x.

Day 4: Number Theory

Topic: The Frobenius Coin Problem

The **Frobenius Coin Problem**, famously known as the "Chicken McNugget Theorem," deals with finding integer solutions to linear combinations. Specifically, it asks: "*What is the largest number that **cannot** be formed by adding multiples of two specific integers?*"

If you have coins worth a cents and b cents, and you have an infinite supply of both, there are some totals you can make (like $a + b$ or $3a$) and some you cannot. If a and b are relatively prime and positive, eventually you reach a point where every integer higher than that point can be formed.

The largest integer that you **cannot** form is called the Frobenius Number.

Questions

1. **Short Answer:** In a video game, you can only score points in bundles of **5** and **7**. What is the largest integer score that is impossible to achieve?

2. **Short Answer:** My McDonald's sells chicken nuggets in small boxes of s nuggets and large boxes of l nuggets, where s and l are relatively prime positive integers with $s < l$. The manager notes that the largest number of nuggets that cannot be purchased by combining these boxes is 53.

   To attract more customers, the manager increases the size of the small box by 2 nuggets. The box sizes remain relatively prime, while the largest number of nuggets that cannot be purchased increases to 71. Find the value of $s + l$.

3. **Proof:** Let a and b be relatively prime positive integers. Find the formula for the largest number that cannot be expressed in the form $ax + by$ in terms of a and b, and prove why this would be the Frobenius Number.

Day 5: Probability

Topic: Derangements

In combinatorics, a **derangement** is a specific permutation of a set of elements where **no** element appears in its original position.

The classic visualization is the "Hat Check Problem": Imagine n people check their hats at a restaurant. The attendant forgets which hat belongs to whom and hands them back completely randomly. A derangement is the scenario where nobody receives their own hat.

To solve this, we cannot simply count the "good" scenarios directly because they are messy. Instead, we use the **Principle of Inclusion-Exclusion**.

- Start with all possible permutations ($n!$).
- Subtract the scenarios where at least 1 person gets their hat back.
- But this subtracts too much (scenarios where 2 people get hats were subtracted twice), so we add those back.
- We continue subtracting and adding ("including and excluding") to reach the exact number.

Questions

1. **Short Answer:** Four friends (A, B, C, D) put their names in a hat and draw them out randomly. In how many scenarios does nobody draw their own name?
2. **Short Answer:** At a math contest with 120 students, 70 students take Algebra, 55 students take Geometry, and 40 students take Number Theory. If 25 students take both Algebra and Geometry, 18 take both Algebra and Number Theory, and 12 take both Geometry and Number Theory, and 5 students take all three subjects, how many students take none of the three subjects?
3. **Proof:** Using the Principle of Inclusion-Exclusion, prove and find the formula for the number of derangements D_n.